J. DIFFERENTIAL GEOMETRY
Vol. **44** (1996) 789-806

ON THE ASYMPTOTIC CONE OF GROUPS SATISFYING A QUADRATIC ISOPERIMETRIC INEQUALITY

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Abstract

We prove that the asymptotic cone of a group satisfying a quadratic isoperimetric inequality is simply connected.

0. Introduction

The asymptotic cone of a group was introduced in [3], where it was used to prove that a group of polynomial growth is virtually nilpotent. It turns out that the group of isometries of the asymptoptic cone of a group of polynomial growth is a Lie group and plays a crucial role in Gromov's proof.

In [1] the construction of the asymptotic cone was generalized to arbitrary finitely generated groups. A complication appears in this case as one has to use ultrafilters in the definition, and it is not clear if the cone depends on the ultrafilter chosen. Because of this sometimes we will refer to all the asymptotic cones of a group as it is not known if this cone is unique. When on the other hand we speak of 'the' asymptotic cone of a group we mean that a specific ultrafilter has been fixed. It is known in many cases (e.g. for hyperbolic groups) that the cone is in fact independent of the ultrafilter .

In [5] Gromov relates the asymptotic cone of a group to the isoperimetric inequalities satisfied by the group. He proves that if every asymptotic cone of a group is simply connected, then the group satisfies a polynomial isoperimetric inequality. A more detailed exposition of this important result has been given by Drutu in [2]. Examples of groups

Received June 13, 1994, and, in revised form, March 6, 1995.

with simply connected asymptotic cones are nilpotent groups (see [7]), hyperbolic groups (in which case the asymptotic cone is an $\mathbb{R}$-tree) , certain solvable groups (see [4]) and combable groups . Gromov in [4] conjectures that the asymptotic cone of $SL_n(\mathbb{Z})$, $n > 3$ is simply connected.

Recently Kapovich and Leeb [5] have used the asymptotic cone of a group to prove that certain groups are not quasiisometric.

Gromov conjectured in [4] that every asymptotic cone of a group satisfying a quadratic isoperimetric inequality is simply connected. In [4] this problem is reduced to proving that groups satisfying a quadratic isoperimetric inequality have a certain metric property. We formulate here (in a slightly different form) Gromov's metric condition:

Let $G =< S|R >$ be a finitely presented group and let $\Gamma_S(G)$ be the Cayley graph of G. Let C be a closed path in $\Gamma_S(G)$. We think of C as a map $f : S^1 \to \Gamma_S(G)$, $S^1 \subset \mathbb{R}^2$, and S^1 is the boundary of a disc D. A collection of discs $D_1, ..., D_p$ is a partition of D if $D = D_1 \cup ... \cup D_p$ and $D_i \cap D_j = \partial D_i \cap \partial D_j$, $1 \leq i, j \leq p$.

A partition of C, which we denote by Π, is a map extending f to ∂D_i, $1 \leq i \leq p$ where $D_1, ..., D_p$ is a partition of D as above. We define the mesh of Π by

$$mesh(\Pi) = \max_{1\leq i\leq p}\{length(\Pi(\partial D_i))\}.$$

Gromov in [4] shows that if there is a k such that every sufficiently long simple closed path C in $\Gamma_S(G)$ can be partitioned into k "pieces" such that the mesh of the partition is less than $length(C)/2$, then every asymptotic cone of G is simply connected. Indeed such a partition induces a similar partition of simple closed curves in each asymptotic cone, and using the fact that an asymptotic cone is a complete metric space (see [1]) one easily sees that every asymptotic cone of G is simply connected. In section 1 we explain this in detail.

In the rest of the paper we show that if G satisfies a quadratic isoperimetric inequality such partitions do exist. In the proof we consider a minimal van Kampen diagram corresponding to a curve in $\Gamma_S(G)$. We first show (sec. 2) that "thin" diagrams can be partitioned, and then (sec. 3) we slice "thick" diagrams into a bounded number of "thin" diagrams by moving the boundary of the diagram in the normal direction (see Figure 1).

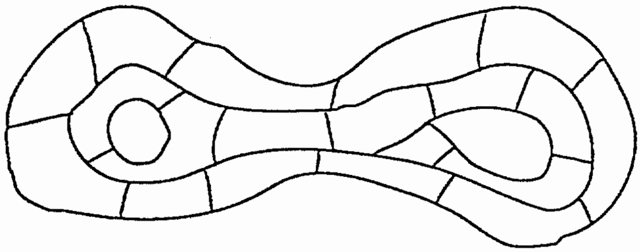

FIGURE 1

1. Preliminaries

We recall some definitions from [4] : A *non-principal ultrafilter* is a finitely additive measure ω defined on all subsets $A \subset \mathbb{N}$, such that

1. $\omega(A)$ equals 0 or 1 for all $A \subset \mathbb{N}$,

2. $\omega(A)$ equals 0 for all finite subsets $A \subset \mathbb{N}$.

Given a bounded function $\phi : \mathbb{N} \to \mathbb{R}$ the *(ultra)limit* of ϕ with respect to ω denoted by $\phi(\omega)$ is the unique real number satisfying the following condition: for every $\epsilon > 0$ the subset $I \subset \mathbb{N}$ where ϕ is ϵ-close to $\phi(\omega)$, i.e.,

$$I = \{i \in \mathbb{N} : |\phi(i) - \phi(\omega)| \leq \epsilon\},$$

has $\omega(I) = 1$.

Let now X be a metric space. We fix $x_0 \in X$ and consider the set of maeps $f : \mathbb{N} \to X$ such that $d(f(i), x_0) \leq c_f i$ for all i, where c_f is a constant. We define the distance of any two such functions f_1, f_2 by $d(f_1, f_2) = \phi(\omega)$ where $\phi(i) = \frac{1}{i} d(f_1(i), f_2(i))$, and ω refers to a chosen non-principal ultrafilter. We define an equivalence relation: $f_1 \equiv f_2$ if and only if $d(f_1, f_2) = 0$. Dividing the set of maps by this equivalence relation we get a metric space called the asymptotic ω-cone of X and denoted by $Con_\omega X$. We have now the following (see also [1]):

Proposition. *$Con_\omega X$ is complete for every ultrafilter ω.*

Proof. Let f_n be a Cauchy sequence in $Con_\omega X$. We define

$$A_k = \{i : |\frac{1}{i} d(f_s(i), f_t(i)) - d(f_s, f_t)| < 1/k, \ \ 1 \leq s, t \leq k\}.$$

Clearly $A_{k+1} \subset A_k$ and $\omega(A_k) = 1$ for all k. We define $k(i) = sup\{k : i \in A_k\}$ if the supremum is not ∞. Otherwise we define $k(i) = i$. Let f be given by:

$$f(i) = f_{k(i)}(i).$$

It is clear that $f = lim_{n\to\infty} f_n$. q.e.d.

Proposition. *If X is a geodesic metric space, then $Con_\omega X$ is a geodesic metric space for every ultrafilter ω.*

Proof. Let $f, g \in Con_\omega X$. Let $c_i : [0,1] \to X$ be geodesic arcs parametrised proportionally to the arc length such that $c_i(0) = f(i)$, $c_i(1) = g(i)$. We define

$$c : [0,1] \to Con_\omega X$$

by $c(t) = \{c_i(t)\}$. It is clear that $c(0) = f$, $c(1) = g$ and that c is a geodesic segment. q.e.d.

Definitions. An *n-gon* S in X is a map from the set of vertices of the standard regular n-gon in the plane into X. We denote the standard regular n-gon by $\overline{S}_n$. We call the edges or sides of S the pairs of points in X corresponding to the pairs of vertices in $\overline{S}_n$ joined by edges. The length of an edge is the distance between the corresponding points. The length of S is the sum of the lengths of its edges. A partition of $\overline{S}_n$ is a collection of discs $D_1, ..., D_k$ such that $\overline{S}_n = \partial(D_1 \cup ... \cup D_k)$ and $D_i \cap D_j = \partial D_i \cap \partial D_j$ when $i \neq j$. We call a point p on $\partial D_1 \cup ... \cup \partial D_k$ a branching point of the partition if for all open sets U containing p, $U \cap (\partial D_1 \cup ... \cup \partial D_k)$ is not homeomorphic to an interval. We call a point a vertex of the partition if it is either a vertex of $\overline{S}_n$ or a branching point. A *partition* of S is a map Π from the set of vertices of a partition of $\overline{S}_n$ to X taking the vertices of $\overline{S}_n$ to S. We call vertices of Π the points in X corresponding to the vertices of the partition of $\overline{S}_n$, and edges of Π the pairs of vertices corresponding to the adjacent vertices of the partition of $\overline{S}_n$. If X is a geodesic metric space we can extend Π to $\partial D_1 \cup ... \cup \partial D_k$ by mapping the arcs between adjacent vertices of the partition of $\overline{S}_n$ to the geodesics joining the points corresponding to those vertices in X. We define the mesh of Π by

$$mesh(\Pi) = \max_{1\leq i\leq k}\{length(\Pi(\partial D_i))\}.$$

Lemma. *A partition $D_1, ..., D_k$ of $\overline{S}_n$ has less than or equal to $n + 2k - 2$ vertices.*

Proof. We see the partition as a planar graph. Let e be the number of edges of this graph and let v be the number of its vertices. If v_1 is the number of vertices corresponding to the branching points, we see that $e \geq \frac{3v_1}{2}$, while $v \leq v_1 + n$. Using Euler's formula we see that $v \leq n + 2k - 2$. q.e.d.

We call two partitions Π_1, Π_2 of an n-gon S equivalent if there is an edge preserving map f from the vertices of Π_1 onto the vertices of Π_2 fixing the vertices of S. $f \circ \Pi_1$ is then a partition of S having the same vertices and the same mesh as Π_2. It is clear that there are finitely many equivalence classes of partitions of S having a fixed number of vertices. In fact, for any k, n there is a finite set $T_{k,n}$ of partitions of $\overline{S}_n$ into k discs such that for any partition of an n-gon S in X into k pieces there is a partition with the same vertices and mesh defined using a partition of $\overline{S}_n$ lying in $T_{k,n}$.

Proposition. *Let X be a metric space. Suppose that for some k every sufficiently long polygon S in X can be partitioned into k pieces of length less than or equal to $length(S)/2$. Then every polygon P in $Con_\omega(X)$ can be partitioned into k pieces of length less than or equal to $length(P)/2$.*

Proof. Let $P = (P_1 ... P_n)$ be an n-gon in $Con_\omega(X)$. Let $P_1^i, ..., P_n^i$ be sequences in X converging (with respect to ω) to $P_1, ..., P_n$. There is a subset of $\mathbb{N}$ with ω measure 1 such that for all i in this set, the polygons $Q_i = (P_1^i ... P_n^i)$ can be partitioned into k pieces of length less than or equal to $length(Q_i)/2$. From the remarks preceding the proposition it follows that there is a set $A \subset \mathbb{N}$ with $\omega(A) = 1$ and a partition of $\overline{S}_n = (S_1 ... S_n)$ with r vertices $A_1, ..., A_r$,where $r \leq n + 2k - 2$, such that for each $i \in A$ there is a partition Π_i of Q_i with r vertices and with $mesh(\Pi_i) \leq length(Q_i)/2$ defined using the given partition of $\overline{S}_n$ and such that $\Pi_i(S_j) = P_j^i$, $j = 1, ..., n$. We define now a partition Π of P using the same partition of $\overline{S}_n$ by $\Pi(A_j) = \{\Pi_i(A_j\}$. Since $\omega(A) = 1$ this is well defined. It is clear that $mesh(\Pi) \leq length(P)/2$. q.e.d.

Proposition. *Let X be a complete metric space. Assume that there is a k such that every n-gon S in X has a partition Π with k pieces such that $mesh(\Pi) \leq length(S)/2$. Then X is simply connected.*

Proof. Let $f : S^1 = \partial D \to X$. We will show how to extend f to $\overline{f} : D \to X$. Let S_n be a sequence of regular 2^n-gons inscribed in S^1 and such that the vertices of S_n are a subset of the vertices of S_{n+1} for all n. Let P_n be the images of S_n under f. Let Π_n be a sequence of partitions of P_n corresponding to finer and finer partitions $D_n^1, ..., D_n^{j_n}$

of S_n such that $mesh(\Pi_n) \leq 1/2^n$ and where each disc $D_n^1, ..., D_n^{j_n}$ is partitioned in exactly k pieces in the $(n+1)$st partition (note that S_n is contained in S_{n+1}). We have the following lemma.

Lemma. *Let x be a vertex of the (n+1)st partition of S_n lying in D_n^j and let y be a vertex of D_n^j. Then $d(\Pi_{n+1}(x), \Pi_{n+1}(y)) \leq k/2^{n+1}$.*

Proof. A simple path consisting of edges of the partition Π_{n+1} joining $\Pi_{n+1}(x)$ to $\Pi_{n+1}(y)$ has at most k vertices and each edge has length less than $1/2^{n+1}$. q.e.d.

Let $x \in D \setminus \partial D$. Let $x \in D_n^{i(x)}$ where $D_n^{i(x)}$ is a disc in the domain of P_n (this makes sense when n is sufficiently large). Let x_n be a vertex of $D_n^{i(x)}$. We then define $\overline{f}(x) = lim_{n \to \infty} \Pi_n(x_n)$. The previous lemma implies that $\Pi_n(x_n)$ is a Cauchy sequence; therefore the limit exists. By the same lemma we see that the limit is independent of the choice of x_n. We will show that $\overline{f}$ is a continuous extension of f. We distinguish two cases:

Case 1. Let $x \in D \setminus \partial D$. Let $\epsilon > 0$ be given, and n be such that x lies in the interior of S_n and $k/2^{n-1} < \epsilon$. Let U be an open ball around x contained in the union of discs in the partition of S_n, which contain x . Then the previous lemma and the definition of $\overline{f}$ imply that $d(\overline{f}(x), \overline{f}(y)) < \epsilon$, for all $y \in U$, i.e., $\overline{f}$ is continuous at x.

Case 2. Let $x \in \partial D$. Let $\epsilon > 0$ be given, and n be such that $k/2^n < \epsilon/2$, and let U be an open ball around x such that for all $y \in U \cap \partial D$, $d(f(x), f(y)) < \epsilon/2$. Assume moreover that U intersects only the discs of the partition of S_n that contain x. Clearly for all $y \in U$ we have $d(\overline{f}(x), \overline{f}(y)) < \epsilon$, and therefore $\overline{f}$ is continuous at x. q.e.d.

2. Thin diagrams

Definitions. We recall the definition of a van Kampen diagram from [6]. A *map* is a finite, planar, connected and simply connected 2-complex. A *diagram* D over an alphabet S is a map such that every edge (i.e., 1-cell) e is provided with a label $\phi(e) \in S$ such that $\phi(e)^{-1} = \phi(e^{-1})$. The label of a path $p = e_1 e_2 ... e_n$ is the word $\phi(e_1)\phi(e_2)...\phi(e_n)$. Call a diagram D over S a *van Kampen diagram* over the group G given by a presentation $< S|R >$ if the label of the boundary path of every face (i.e., 2-cell) of D is a cyclic permutation of some relator $r^{\pm 1} \in R$. The length, $l(p)$, of a path p in a diagram is equal to the number of edges of the path. The boundary of a van Kampen diagram D, denoted by ∂D, is a closed path of minimal length which contains all the edges of D not lying in the interior of D. Note that our definition is slightly more general than the one given in [6], namely, we do not require that the label of the boundary of D is a reduced word. For example a path p labelled by a word w can be considered also as a van Kampen diagram having as boundary label ww^{-1}. This more general definition of van Kampen diagrams does not cause any problems and is more convenient for our purpose.

Let w be a word in the alphabet S. Then w represents the identity in G if and only if there is a van Kampen diagram over G such that w is the boundary label of D. A *minimal van Kampen diagram* for a word w is a van Kampen diagram with boundary label w and the minimum possible number of faces. The *area* of a word w, $A(w)$, is the area of a minimal van Kampen diagram D with boundary label w which is, by definition, the number of faces (2-cells) of D.

The *length*, $l(w)$, of a word w is the number of letters in the word. We denote by $\overline{K}$ the closure of a subcomplex K of D . We define $star(K)$ to be the set of all closed cells which intersect K, and denote by $star_i(K)$ the subcomplex of D obtained by iterating the star operation i times. If P is a vertex of D, we define the ball of radius r and center P to be: $B_P(r) = star_r(P)$. Note that for every vertex $Q \in B_P(r)$, $d(P,Q) \leq r$. We define the sphere of radius r around P to be $S_P(r) = \overline{D - B_P(r)} \cap B_P(r)$, and the distance, $d(P,Q)$, between two vertices P, Q on D to be the length of the shortest path in D joining them. If P, Q are on ∂D we define $d_\partial(P,Q)$ to be the length of the shortest path on ∂D joining them.

We define the radius of a van Kampen diagram D to be $r(D) = max\{d(x, \partial D) : x \text{ is a vertex of } D\}$. Let D be a van Kampen diagram, and ∂D be its boundary. Let $f : S^1 \to \partial D$ be a parametrization of ∂D with respect to the arc length, where S^1 is the circle of length $l = l(\partial D)$. If $t_1, t_2 \in S^1$ we denote by $\overline{t_1 t_2}$ the arc of S^1 having as initial point t_1 and as terminal point t_2 if we orient S^1 in the counterclockwise direction. $f(\overline{t_1 t_2})$ is then a subpath of ∂D with endpoints $f(t_1), f(t_2)$. If $P = f(t_1), Q = f(t_2)$ are two vertices of ∂D, we will abuse notation and write $\overline{PQ}$ instead of $f(\overline{t_1 t_2})$. So $\overline{PQ}$ is an oriented subpath of ∂D with initial vertex P and terminal vertex Q if we orient ∂D in the counterclockwise direction. Note that if ∂D intersects itself, $\overline{PQ}$ is not always well defined. If either P or Q is point of self-intersection of ∂D, there is more than one path satisfying the definition of $\overline{PQ}$. In such situations when we write $\overline{PQ}$ it means that we choose arbitrarily any of the oriented paths with initial point P and terminal point Q. Note however that as soon as we choose a path $\overline{PQ}$, the path $\overline{QP}$ is well defined: it is the complement path of $\overline{PQ}$ (i.e., $\overline{PQ} \cup \overline{QP} = \partial D$). In the rest of the paper we will follow this convention.

Given a finite presentation $< S|R >$ of a group G we can 'triangulate' it as follows: If some $r \in R$ has length more than 3, then $r = ab$ for some words a, b of length more than 1. Introduce a new generator x and observe that $< S \cup \{x\} | (R - \{r\}) \cup \{xa^{-1}, xb\} >$ is also a presentation of G. Repeating this step finitely many times we arrive at a triangular presentation of G, i.e., a presentation in which every relator has length at most 3.

Proposition. *Let D be a van Kampen diagram with $r(D) \leq l(\partial D)/25$. Let $m \in \mathbb{N}, m > 0$, be such that $r(D) \leq m \leq l(\partial D)/25$. Then $D = D_1 \cup D_2$, where D_1, D_2 are van-Kampen diagrams, $D_1 \cap D_2$ is a simple path, and*

1. $l(\partial D_1) \leq 25m$,
2. $l(\partial D_2) \leq l(\partial D) - m$.

Proof.

Lemma. *Let D be a van Kampen diagram. Let P, Q be vertices on ∂D and let α be a simple path on $D^{(1)}$ joining them with*

$$l(\alpha) \leq min(\overline{PQ}, \overline{QP}) - m.$$

Then α induces a partition of D in two van Kampen diagrams D_1, D_2 such that: $D = D_1 \cup D_2$, $D_1 \cap D_2 = \alpha$ and $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$.

Proof. Indeed if D_1 is the subdiagram of D with boundary $\alpha \cup \overline{PQ}$, and D_2 is the subdiagram of D with boundary $\alpha \cup \overline{QP}$, we have $D = D_1 \cup D_2$, $D_1 \cap D_2 = \alpha$ and $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$. q.e.d.

We return now to the proof of the proposition.We distinguish 2 cases:

Case 1. Suppose that for every vertex P in ∂D there is a simple path α in $D^{(1)}$, with initial vertex P and $l(\alpha) \leq 4m$ separating D in two van Kampen diagrams D_1, D_2 such that $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$.

We claim that under this hypothesis the proposition is true. Among all simple paths of length less or equal to $4m$ separating D in D_1, D_2 such that $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$ we pick a path α for which $l(\partial D_1)$ attains its minimal value. If $l(\partial D_1) \leq 25m$ the partition of D in D_1, D_2 by α satisfies the requirements of the above proposition and we are done. We assume therefore that $l(\partial D_1) > 25m$. We have $\partial D_1 = \overline{PQ} \cup \alpha$, where P, Q are the endpoints of α and $\overline{PQ}$ is a subpath of ∂D. Since $l(\alpha) \leq 4m$ we have $l(\overline{PQ}) > 21m$. Therefore there is a point $Q_1 \in \overline{PQ}$ with $l(\overline{PQ_1}) = 13m$. By the hypothesis of *case* 1 there is a path β with initial vertex Q_1 and $l(\beta) \leq 4m$ separating D in two diagrams D_1', D_2' such that

$$l(\partial D_i') \leq l(\partial D) - m, \quad i = 1, 2.$$

By our minimality assumption for α the endpoint of β does not lie on $\overline{PQ}$ hence β intersects α at a point Q_2. Therefore

$$d(P, Q_1) \leq d(P, Q_2) + d(Q_2, Q_1) \leq l(\alpha) + l(\beta) \leq 8m.$$

Then a geodesic path γ joining P to Q_1 separates D into two diagrams D_1", D_2" which by the above lemma satisfy the inequalities $l(\partial D_i\text{"}) \leq l(\partial D) - m$, $i = 1, 2$. Moreover the boundary of one of the two diagrams, say D_1", is $\gamma \cup \overline{PQ_1}$. Therefore $l(\partial D_1\text{"}) \leq 21m < 25m$, i.e., D_1", D_2" give the required partition in this case.

<u>*Case* 2.</u> We assume that the assumption of *case* 1 is not valid, i.e., we assume that there is a vertex $P \in \partial D$ such that there is no simple path α with $\alpha(0) = P$ and $l(\alpha) \leq 4m$ separating D in two diagrams D_1, D_2 with $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$. We will show that in this case the proposition is also true.

We consider $B = B_P(3m)$. Suppose that there is $Q \in B \cap \partial D$ such that $min(\overline{PQ}, \overline{QP}) > 4m$. If α is the geodesic path joining P to Q, then we have $l(\alpha) \leq 3m$ and by the above lemma α separates D in two diagrams D_1, D_2 with $l(\partial D_i) \leq l(\partial D) - m$, $i = 1, 2$ which contradicts the hypothesis of *case* 2. Therefore for all $Q \in B \cap \partial D$ either $l(\overline{PQ}) \leq 4m$ or $l(\overline{QP}) \leq 4m$. Hence there are vertices $P_1, P_2 \in S_P(3m) \cap \partial D$ such that the following hold:

1. $l(\overline{P_1P}) \leq 4m$, $l(\overline{PP_2}) \leq 4m$.
2. For all $Q \in B \cap \partial D$ with $l(\overline{QP}) \leq 4m$ we have $\overline{QP} \subset \overline{P_1P}$, and for all $Q \in B \cap \partial D$ with $l(\overline{PQ}) \leq 4m$ we have $\overline{PQ} \subset \overline{PP_2}$.

Clearly $6m \leq l(\overline{P_1P_2}) \leq 8m$. Let p be a path in $S_P(3m)$ connecting P_1 to P_2. To see that there is such a path consider the connected component of $D - B$ containing $\overline{P_1P_2}$. Let C be the closure of this connected component. Then C is a van Kampen diagram and $\partial C = \overline{P_1P_2} \cup p$ where p is a simple path in $S_P(3m)$ connecting P_1 to P_2.

For every vertex $Q \in p$ we pick $Q^0 \in \partial D$ such that $d(Q, Q^0) = d(Q, \partial D)$. By our hypothesis that $r(D) \leq m$ we have $d(Q, Q^0) \leq m$.

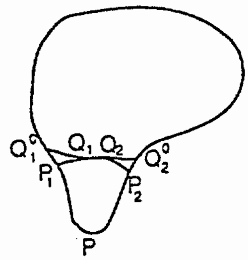

FIGURE 2

Since $d(P,Q) = 3m$, $4m \geq d(P,Q^0) \geq 2m$. We claim that for all Q^0, $min(l(\overline{PQ^0}), l(\overline{Q^0P})) \leq 5m$. Indeed if for some Q^0 this is not true, it follows from the inequalities and the above lemma that the geodesic path joining P to Q^0 separates D in two van Kampen diagrams D_1, D_2 with $l(\partial D_i) \leq l(\partial D) - m$, $i = 1,2$ which contradicts the hypothesis of *case* 2.

Therefore for all Q^0 we have $d_\partial(Q^0, P_1) \leq 2m$ or $d_\partial(Q^0, P_2) \leq 2m$. Hence there are vertices Q_1, Q_2 on p with $d(Q_1, Q_2) \leq 1$ and $d_\partial(Q_1^0, P_1) \leq 2m$, $d_\partial(Q_2^0, P_2) \leq 2m$. But then $d(P_1, P_2) \leq 6m + 1$. On the other hand $8m \geq l(\overline{P_1P_2}) \geq 6m$. If α is a geodesic path joining P_1 to P_2,then,by the lemma above, α separates D in two diagrams D_1, D_2. Moreover $\partial D_1 = \alpha \cup \overline{P_1P_2}$ and $\partial D_2 = \alpha \cup \overline{P_2P_1}$. Clearly $l(\partial D_1) \leq 25m$, $l(\partial D_2) \leq l(\partial D) - m$. Therefore in this case too there is a partition of D in D_1, D_2 with the required properties. This finishes the proof of the proposition. q.e.d.

Corollary. *Let D be a van Kampen diagram. Let $m > 0$ be such that $r(D) \leq m$. Then $D = D_1 \cup ... \cup D_k$ where $D_i, i = 1, ..., k$ are subdiagrams of D , $D_i \cap D_j, (0 \leq i,j \leq k)$ is empty or a vertex or a simple path, $l(\partial D_i) \leq 25m$ and $k \leq \frac{l(\partial D)}{m} + 1$.*

Proof. If $l(\partial D) \leq 25m$. Then the assertion above is clearly true. Otherwise it follows by induction on $l(\partial D)$ using the proposition above.

q.e.d.

3. The general case

Let G be a group given by a triangular presentation $< S|R >$ satisfying a quadratic isoperimetric inequality $A(w) \leq Ml(w)^2$ where

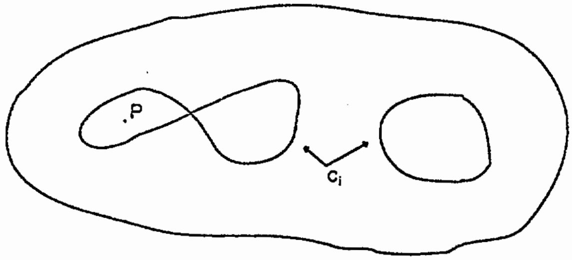

FIGURE 3

$M \in BbbN$. We define the filling radius function of G by

$$f(n) = \max_{l(w) \leq n} \min\{r(D)\,|\partial D = w\}.$$

Note that this definition is slightly different than the one in [4]. Let us assume that R contains all the words of length less than or equal to 3 which are trivial in G. We have then:

Proposition. $f(n) \leq 12Mn$.

Proof. We will prove by induction on n that if D is a minimal van Kampen diagram for a word of length n, then $r(D) \leq 12Mn$. For $n \leq 3$ it is obviously true. Let w be a word on S with $l(w) = n$. Let D be a minimal van Kampen diagram for w.

We define $N_i = star_i(\partial D)$, $1 \leq i \leq 6Mn - 1$. If $c_i = \partial N_i - \partial D$, then $A(N_{i+1}) - A(N_i) \geq l(c_i)/3$. This is because each 1-cell of c_i lies in the boundary of a 2-cell in $N_{i+1} - N_i$. If $l(c_i) > n/2$ for all $1 \leq i \leq 6Mn$, then

$$A(D) > \frac{n}{2}\frac{1}{3}6Mn > Mn^2,$$

which is impossible. Therefore $l(c_i) \leq n/2$ for some i, $1 \leq i \leq 6Mn$. We note now that c_i is a union of simple closed curves any two of which are either disjoint or intersect at exactly one point. Every vertex P of D either lies in the interior of some simple closed curve of c_i or is of distance less than or equal to $6M$ from ∂D. If P lies in the interior of some simple closed curve of c_i, then by the inductive hypothesis $d(P, c_i) \leq 6Mn$, so $d(P, \partial D) \leq 12Mn$.

Theorem. *There is a k such that for every minimal van Kampen diagram D of G with $l(\partial(D)) \geq 200$ we have that $D = C_1 \cup \ldots \cup C_k$ where $C_i, i = 1, \ldots, k$, are van Kampen subdiagrams of D , $C_i \cap C_j, 0 \leq i, j \leq k$,* *is empty or a vertex or a simple path and* $l(\partial C_i) \leq l(\partial D)/2$ *for all* $i = 1, ..., k$.

Remark. We show in the proof that we can take $k = 120 \cdot 600^3 \cdot M^4$ but this is far from the best estimate for k.

Proof. Let us assume that $l(\partial D) = n$. We decompose D into a union of 'annuli':

Let $B_1' = star_i(\partial D)$ where i is such that:

$$n/200 \leq i \leq n/100$$

and

$$l(\partial(star_i(\partial D)) - (\partial D)) \leq 600Mn.$$

Such an i exists because if

$$l(\partial(star_i(\partial D)) - (\partial D)) > 600Mn$$

for all

$$n/200 \leq i \leq n/100,$$

then

$$A((star_{n/100}(\partial D)) > 600Mn\frac{n}{200}\frac{1}{3} > Mn^2.$$

We define:

$$B_1 = B_1' \cup \{\overline{C} | \, C \text{ conn. comp. of } D - B_1' \text{ with } l(\partial \overline{C}) < \frac{n}{1200M}\}.$$

Let $D_1 = B_1$. Let $B_2' = star_i(B_1)$ where i is such that

$$n/200 \leq i \leq n/100$$

and

$$l(\partial(star_i(\partial B_1)) - (\partial D)) \leq 600Mn.$$

Let

$$B_2 = B_2' \cup \{\overline{C} | \, C \text{ conn. comp. of } D - B_2' \text{ with } l(\partial \overline{C}) < \frac{n}{1200M}\}.$$

Let $D_2 = \overline{B_2 - B_1}$ and inductively:

$$B_{r+1}' = star_i(B_r),$$

where i is such that

$$n/200 \leq i \leq n/100$$

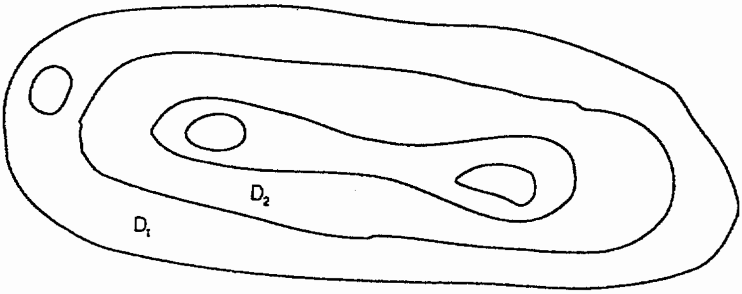

FIGURE 4

and

$$l(\partial(star_i(\partial B_r)) - (\partial D)) \leq 600Mn.$$

Let

$$B_{r+1} = B'_{r+1} \cup \{\overline{C} | C \;\; conn. \;\; comp. \;\; of \quad D - B'_{r+1} \quad with \quad l(\partial\overline{C}) < \frac{n}{1200M}\}.$$

Let $D_{r+1} = \overline{B_{r+1} - B_r}$. The sequence terminates when

$$D = D_1 \cup D_2 \cup ... \cup D_p.$$

Since $r(D) \leq 12Mn$, we have

$$p \leq \frac{12Mn}{\frac{n}{200}} = 2400M.$$

We will show that each 'annulus' $D_r, r = 1, ..., p$, can be decomposed into less than $42 \cdot 10^4 \cdot M^2$ pieces such that the length of the boundary of each piece is less than $n/2$. $\partial D_r - \partial D_{r-1}$ has length less than $600Mn$ and it is a union of simple closed paths. By the definition of D_r each of these simple closed paths has length more than $\frac{n}{1200M}$. We conclude that $\partial D_r - \partial D_{r-1}$ can be written as a union of at most

$$\frac{600Mn}{\frac{n}{1200M}} = 72 \cdot 10^4 M^2$$

simple closed paths. For each of these paths we pick a (simple) path of length less than $n/100$ joining it to $\partial D_r \cap \partial D_{r-1}$. We cut D_r open along these paths and get a diagam A_r which is a union of van Kampen diagrams and

$$l(\partial A_r) \leq 2 \cdot 600Mn + 72 \cdot 10^4 M^2 \cdot \frac{n}{100} \leq 14 \cdot 600M^2 \cdot n.$$

Each connected component of A_r has radius less than $n/50$; therefore we can apply the corollary of the previous section to decompose it to pieces of boundary length less than $n/2$. In fact a component of boundary length l can be decomposed into less than $\frac{50l}{n}+1$ pieces of boundary length less than $n/2$. On the other hand each connected component of A_r has length more than $\frac{n}{1200M}$ so A_r has at most

$$\frac{14 \cdot 600M^2 n}{\frac{n}{1200M}} = 28 \cdot 600^2 M^3$$

components. If a component has boundary length less than $n/2$, then we leave it as it is; otherwise, we decompose it using the corollary of the previous section. It is clear that A_r can be decomposed into less than $(50 \cdot 14 \cdot 600M^2 \cdot n)/n + 28 \cdot 600^2 M^3 \leq 30 \cdot 600^2 M^3$ pieces of boundary length less than $n/2$, and therefore D_r can be decomposed into less than $30 \cdot 600^2 M^3$ pieces of boundary length less than $n/2$. Since $D = D_1 \cup ... \cup D_p$, D can be decomposed into less than

$$30 \cdot 600^2 M^3 \cdot 4 \cdot 600M = 120 \cdot 600^3 \cdot M^4$$

pieces of boundary length less than $n/2$.

4. A more refined estimate

In this section we refine the results of section 3 proving a stronger decomposition theorem for van Kampen diagrams for groups satisfying a quadratic isoperimetric inequality. In what follows we assume as in section 3 that G is a group given by a triangular presentation $< S|R >$ satisfying a quadratic isoperimetric inequality $A(w) \leq Ml(w)^2$ and we consider van Kampen diagrams over G.

Let us denote by $M(a)$ the minimal number such that any van Kampen diagram of length n (where n is large enough) can be decomposed into $M(a)$ pieces of boundary length less or equal to n/a. Gromov then conjectures (see [4], $5F_2$) that

$$liminf_{a\to\infty} \frac{log(M(a))}{log(a)} \leq 2.$$

To see what this says note that one can subdivide a square of side length 1 into 4^k equal squares of side length $1/2^k$ in the obvious way. One can modify the proof of the previous section and prove this conjecture.

More precisely we will show that there is a $K > 0$ such that for any $a > 0, a \in \mathbb{N}$ one can decompose a minimal van Kampen diagram with boundary length $n > 100a$ into less than Ka^2 pieces such that the length of the boundary of each piece is less than or equal to n/a.

We remark that if all minimal van Kampen diagrams over a group satisfy this condition, then the group satisfies a quadratic isoperimetric inequality. In particular it is a stronger condition than the simple connectivity of the asymptotic cone of a group. This clearly implies Gromov's conjecture; it shows in fact that

$$limsup_{a\to\infty}\frac{log(M(a))}{log(a)} \leq 2.$$

The proof is essentially the same as the proof of the theorem in sec.3, the only difference being that we bound $\sum_{r=1}^{p} l(\partial D_r - \partial D_{r-1})$ by cn for an appropriate constant c and we think of subdividing all "annuli" at once. We repeat the construction as we now have to keep track of the dependence of the constants appearing from a.

Let $B_1' = star_i(\partial D)$ where i is such that:

$$n/100a \leq i \leq n/50a,$$

and for which $l(\partial(star_i(\partial D)) - (\partial D))$ takes its minimum value. We define:

$$B_1 = B_1' \cup \{\overline{C} | \, C \text{ conn. comp. of } D - B_1' \text{ with } l(\partial\overline{C}) < \frac{n}{600aM}\}.$$

Let $D_1 = B_1$. Let $B_2' = star_i(B_1)$ where i is such that

$$n/100a \leq i \leq n/50a,$$

and for which $l(\partial(star_i(\partial B_1)) - (\partial D))$ takes its minimum value. Let

$$B_2 = B_2' \cup \{\overline{C} | \, C \text{ conn. comp. of } D - B_2' \text{ with } l(\partial\overline{C}) < \frac{n}{600aM}\}.$$

Let $D_2 = \overline{B_2 - B_1}$, and inductively:

$$B_{r+1}' = star_i(B_r),$$

where i is such that

$$n/100a \leq i \leq n/50a$$

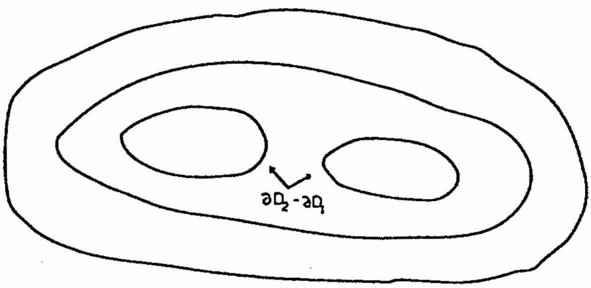

FIGURE 5

and for which $l(\partial(star_i(\partial B_r)) - (\partial D))$ takes its minimum value. Let

$$B_{r+1} = B'_{r+1} \cup \{\overline{C} \,|\, C \; conn.\; comp.\; of \;\; D - B'_{r+1} \;\; with \;\; l(\partial\overline{C}) < \frac{n}{600aM}\}.$$

Let $D_{r+1} = \overline{B_{r+1} - B_r}$. The sequence terminates when

$$D = D_1 \cup D_2 \cup ... \cup D_p.$$

Since $r(D) \leq 12Mn$, we have

$$p \leq \frac{12Mn}{\frac{n}{100a}} = 1200aM.$$

We note now that

$$\sum_{r=1}^{p} l(\partial D_r - \partial D_{r-1}) \leq 300aMn,$$

where we take $\partial D_0 = \partial D$.

Indeed, since by hypothesis $A(D) \leq Mn^2$ and as we have seen earlier, $A(star(D_r)) - A(D_r) \geq \frac{1}{3} l(\partial D_r)$ we have that

$$\sum_{r=1}^{p} l(\partial D_r - \partial D_{r-1}) \frac{1}{3} \frac{n}{100a} \leq Mn^2,$$

where each term in this sum is a lower bound of the area of an "annulus" in D and all these "annuli" are disjoint. Hence

$$\sum_{r=1}^{p} l(\partial D_r - \partial D_{r-1}) \leq 300aMn.$$

We note now that $\partial D_r - \partial D_{r-1}$, $r = 1, ..., p$ is a union of simple closed paths each of which has length at least $n/600aM$. Using the previous inequality we conclude that $\bigcup_{r=1}^{p} (\partial D_r - \partial D_{r-1})$ can be written as a union of less than $\frac{300aMn}{\frac{n}{600aM}} = 18 \cdot 10^4 M^2 a^2$ simple closed paths. For each such closed path lying in $\partial D_r - \partial D_{r-1}$ we pick a simple path of length less than $n/50a$ joining it to ∂D_{r-1} and we cut D_r open along this new simple path. After we do this for each closed path in each D_r we get a collection of van Kampen diagrams A_n, $n = 1, ..., q$, such that for every n:

(1) $r(A_n) \leq \frac{n}{50aM}$,

(2) $l(\partial A_n) \geq \frac{n}{600aM}$,

(3) $\sum_{n=1}^{q} l(\partial A_n) \leq 18 \cdot 10^4 M^2 a^2 \cdot \frac{n}{50aM} + (300aM + 1)n \leq Ean$,

where E in (3) is an appropriately chosen constant depending only on M. By (2) and (3) we see that

$$q \leq \frac{Ean}{\frac{n}{600aM}} \leq 600EMa^2.$$

Using the corollary of section 2 we can decompose $\bigcup_{n=1}^{q} A_n$ into less than

$$\frac{Ean}{\frac{n}{50a}} + 600EMa^2 = (50E + 600EM)a^2$$

pieces of boundary length less than n/a.

5. Final remarks

It is easy to see that if every asymptotic cone of a group is simply connected, then the filling radius grows linearly. Indeed (see [4],[2]) if every asymptotic cone of G is simply connected, then there is a k such that any minimal van Kampen diagram D of G with $l(\partial D) = n$ (where n is large enough) can be decomposed into k pieces such that the length of the boundary of each piece is less than $n/2$. Since any vertex of D is in some such piece, any vertex can be connected to the boundary of the corresponding piece and then to the boundary of D by a path contained in the boundary of pieces (see picture). Therefore the filling radius function of G satisfies:

$$f(n) \leq f(\frac{n}{2}) + k\frac{n}{2},$$

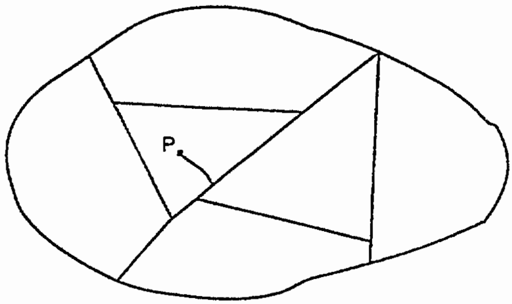

FIGURE 6

which clearly implies that f is bounded by a linear function. This observation makes it natural to ask: Are there groups satisfying a polynomial isoperimetric inequality whose filling radius grows faster than linearly?

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